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ON THE QUESTION OF FORMULATING THE PROBLEM OF OPTIMAL DESIGN OF FREELY OSCILLATING STRUCTURES

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The correctness of the formulation of the problem of minimizing (maximizing) the volume of an elastic structure is discussed under the condition that the form of the free component of the boundary of the body is used as the control, and the fundamental oscillation frequency together with the clamped component of the boundary are fixed. It is shown that the procedure used by Prager yields the necessary and sufficient conditions for the volume of the body to be maximum, and the problem of minimum volume has a trivial (zero) solution unless additional constraints are imposed. A special form of these constraints is proposed.

The problem in question was studied by a number of workers. Prager [1] proposed a method of determining the necessary and sufficient conditions for the volume of the body to be extremal, for a wide class of self-conjugate problems of structure optimization. Armand [2] applied the Prager's method to problems of design of plates of variable thickness, using the latter parameter as a control. Brach [3] had obtained the necessary conditions for a minimum volume of a beam using its thickness as the control. A short exposition of the Prager's method follows.

An optimal body Σ of volume V is bounded by the surface $S = S' \cup S'' \cup S'''$, where the segment S' is under external load, segment S'' is load-free and segment S''' is clamped. The segments S' and S''' are assumed to be fixed, and the segment S'' may be varied. The body Σ_* introduced for purposes of comparison is bounded by the surface $S_* = S' \cup S_*'' \cup S'''$, and its volume

$$V_* = V + V_+ - V_- \tag{1}$$

is obtained by adding to the volume V a volume V_+ and subtracting a volume V_-. The functional

$$c = \min_w \int_{V_*} F_* dV_* \tag{2}$$

where c is a specified constant, w is the displacement of the points of the body and F_* is a specified function of w and its derivatives, is defined on all admissible bodies Σ_*.

We require to find among these admissible bodies a body of minimum (or maximum) volume. According to the condition (2) we have

$$\int_V F\, dV = \int_{V_*} F_* dV_* \tag{3}$$

where the function F is defined in Σ. Taking into account (2), we can write

$$\int_{V_*} F_* dV_* \leqslant \int_{V_*} F\, dV_* = \int_{V} F\, dV + \int_{V_+} F\, dV_+ - \int_{V_-} F\, dV_- \tag{4}$$

from which, using (3) we obtain

$$\int_{V_+} F\, dV_+ - \int_{V_-} F\, dV_- \geqslant 0 \tag{5}$$

Let us set

$$\begin{aligned} F &= F_0 = \text{const} \quad \text{on } S'' \\ F &= F_0 - F_+ \ \text{ in } V_+;\ F_+ \geqslant 0 \\ F &= F_0 - F_- \ \text{ in } V_-;\ F_- \geqslant 0 \end{aligned} \tag{6}$$

From (5) and (6) we obtain

$$F_0(V_+ - V_-) \geqslant \int_{V_+} F_+ dV_+ + \int_{V_-} F_- dV_- \geqslant 0$$

and now, using (1) we can conclude that

$$F_0\,(V_+ - V_-) \geqslant 0 \tag{7}$$

We note that the Prager's method cannot be used when the clamped part of the boundary is used as the control. Indeed, although the Prager's method does not make explicit use of the conditions imposed on the free and clamped parts of the boundary, the inequality (4) holds only for the functions w which satisfy the same basic boundary conditions in Σ_*, as in Σ. For this reason the method cannot be used when the surface on which the basic boundary conditions are specified is varied.

Prager has proposed, in particular, that the method given above be used to determine a body of extremal volume with a specified fundamental fequency ω. In this case F is equal to the density of the Lagrangian of the body

$$F = G - \omega^2 H \tag{8}$$

where G and H denote the densities of the potential and kinetic energy, and the loaded surface segment S' is absent.

The same approach was used later by Armand [2] in the problem of oscillations of a plate of minimum volume. He found that the volume became zero unless a hypothesis of non-structural mass was introduced. Brach [3] has also found that the volume of a beam becomes zero in the absence of concentrated masses lying on it.

The problem of optimal design of freely oscillating bodies exhibits a number of features distinguishing it from the static problems. Firstly, the problem is homogeneous and the values of F and F_0 can be determined to within an arbitrary positive multiplier. This multiplier can be regarded as a coefficient of proportionality connecting the state variables and the Lagrangian multipliers if the problem is solved by the usual variational methods. Secondly, in contrast with the static problems, the quantity F can assume values of different signs.

We shall show that the Prager's method yields the necessary and sufficient conditions for the maximum volume of the body when the fundamental frequency is fixed.

Indeed, the relation (2) is equivalent to the Rayleigh's variational principle

$$\omega^2 = \min_w \frac{\int_V G\,dV}{\int_V H\,dV}$$

or

$$c = \min_w \int_V (G - \omega^2 H)\,dV \equiv 0 \tag{9}$$

According to (6) the boundary value F_0 of the function $F = G - \omega^2 H$ should be minimal in the optimal body Σ. From (9) it follows that

$$F_0 \leqslant 0 \tag{10}$$

and the equality is attained only when $F \equiv 0$ over the whole volume, which corresponds to the stress-free state. Excluding this case from our discussion, we rewrite (10) in the form

$$F_0 = G - \omega^2 H < 0 \tag{11}$$

The inequality (7) enables us to conclude that $V \geqslant V_*$, i.e. the maximum volume structure corresponds to conditions (6).

The condition of optimality (11) implies, in particular, that the optimal body cannot have a connected component of the boundary consisting of a clamped and a free surface. Indeed, from (11) and the positiveness of G it follows that

$$H \geqslant -F_0 / \omega^2 > 0 \tag{12}$$

on the whole surface S''. But the kinetic energy H is a continuous function of the surface element and the condition $H = 0$ holds on S''', therefore the condition (12) cannot hold near S'''.

A body satisfying the Prager conditions "envelops" the clamped boundary and is bounded externally by the free surface.

Example. Let the clamped component of the boundary be a sphere. Then, as we can verify directly, the spherical layer satisfies the conditions (6) [4] and does therefore represent a body of maximum volume with a specified natural frequency and a clamped surface.

Simple physical considerations show that there are no solutions of the converse problem on a body of minimum volume with a specified natural frequency and a clamped part of the boundary S''' unless additional constraints are imposed. Indeed, bringing the free segment of the boundary S'' infinitely near to the clamped part S''', we obtain a body with an arbitrarily large natural frequency of oscillations. Now attaching to this body a long, thin flange resting on S'' and oscillating like a cantilever, we can reduce the natural frequency of the body (which does not exceed the natural frequency of the flange) by as much as we like and, in particular, make it equal to the prescribed frequency. The volume of the flange as well as the volume of the whole body can be made as small as we like, and this agrees with the results of [2, 3].

The authors of [2, 3] have succeeded in obtaining a nontrivial solution of the problem of minimal volume, using the hypothesis of non-structural mass [2] or by introducing concentrated masses [3]. We shall show that under the general formulation of the problem introduction of such an assumption will also enable us to obtain a non-trivial solution.

Let the body Σ have a part Σ_n not subject to variation and consisting of a material with zero potential energy of deformation (e.g. a perfectly rigid body), and a nonzero kinetic energy. Denoting the volume of the remaining Σ by V_v, we can write the equation (9) in the form

$$0 = \min_w \left\{ \int\limits_{V_v} (G - \omega^2 H)\, dV_v - \int\limits_{V_n} \omega^2 H\, dV_n \right\}$$

or

$$\int\limits_{V_v} (G - \omega^2 H)\, dV_v = \omega^2 \int\limits_{V_n} H\, dV_n > 0$$

In this case the inequality

$$F_0 > 0 \tag{13}$$

leading to the condition of minimal volume no longer contradicts the variational Rayleigh principle, and the Prager conditions (6) together with the inequality (13) yield the necessary and sufficient conditions for the minimal volume of the body.

Introduction of the hypothesis of non-structural mass makes possible the regularization of the problems in which the Lagrangian is homogeneous with respect to the control, e.g. the problems discussed in [2, 3]. However, the hypothesis cannot regularize such problems as the problem of distribution of thickness in a Kirchhoff plate of minimum volume with a fixed fundamental characteristic frequency where the absence of the optimal solution is related, as shown in [5], to the fact that all displacements have the same direction.

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REFERENCES

1. Prager, W., Optimality criteria in structural design. Proc. National Acad. Natur. Sci. USA. Vol. 61, No. 3, 1968.

2. Armand, J.-L., Applications of the theory of optimal control of distributed-parameter systems to structural optimization. NASA, CR-2044, 1972.

3. Brach, R. M., On optimal design of vibrating structures. J. Optimizat. Theory and Appl., Vol. 11, No. 6, 1973.

4. Love, A. E. H., A Treatise on the Mathematical Theory of Elasticity. N. Y. Dover, 1944.

5. Lur'e, K. A. and Cherkaev, A. V., Application of the Prager's theorem to the problem of optimal design of thin plates. Izv. Akad. Nauk SSSR, MTT, No. 6, 1976.

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